# Exercise 8

Use the *successive approximations method* to solve the following Volterra integral equations:

$$u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$$

---

### Solution

The successive approximations method, also known as the method of Picard iteration, will be used to solve the integral equation. Consider the iteration scheme,

$$\begin{aligned} u_{n+1}(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_n(t)\,dt, \quad n \geq 0 \\ &= 1 - \frac{1}{2}x^2 + \int_0^x \int_0^q \int_0^r \int_0^s u_n(t)\,dt\,ds\,dr\,dq, \end{aligned}$$

choosing $u_0(x) = 0$. Then

$$\begin{aligned} u_1(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_0(t)\,dt = 1 - \frac{1}{2}x^2 \\ u_2(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_1(t)\,dt = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \\ u_3(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_2(t)\,dt = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 - \frac{1}{3\,628\,800}x^{10} \\ u_4(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_3(t)\,dt = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40\,320}x^8 - \frac{1}{3\,628\,800}x^{10} \\ &\qquad + \frac{1}{479\,001\,600}x^{12} - \frac{1}{87\,178\,291\,200}x^{14} \\ &\vdots, \end{aligned}$$

and the general formula for $u_{n+1}(x)$ is

$$u_{n+1}(x) = \sum_{k=0}^{2n+1} \frac{(-1)^k}{(2k)!}x^{2k}.$$

Take the limit as $n \to \infty$ to determine $u(x)$.

$$\begin{aligned} \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \sum_{k=0}^{2n+1} \frac{(-1)^k}{(2k)!}x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} \\ &= \cos x \end{aligned}$$

Therefore, $u(x) = \cos x$.